The Euclidean Parallel Postulate

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1 Introduction

In Euclid's geometry, there are five axioms that lay a solid foundation for all the theorems and propositions. Euclid's fifth axiom is the parallel postulate. Because people believed that the fifth axiom seemed more like a proposition than an axiom, they tried to prove it from the other four axioms. For centuries, they never succeeded. Finally, in 1868, the Italian mathematician, Eugenio Beltrami, proved the independence of the parallel postulate. Moreover, by using the negation of the Euclidean parallel postulate, people found other geometries such as the hyperbolic geometry of the Russian mathematician Nikolai Ivanovich Lobachevsky and the spherical geometry of the German mathematician Georg Friedrich Bernhard Riemann.

2 Neutral Geometry

As mentioned above, the fifth axiom of Euclidean Geometry cannot be derived from the first four axioms. Let us consider the axiomatic system containing just the first four axioms. People call such a geometry neutral geometry or absolute geometry.

2.1 Axioms and Definitions

Here are first four of Euclid's axioms, which makes up the neutral geometry system.
Axiom 1. We may draw a straight line between any two points.
Axiom 2. We may extend any terminated straight line indefinitely.
Axiom 3. We may draw a circle with any given point as center and any given radius.
Axiom 4. All right angles are equal.
We also make some definitions in our axiom system.
Definition: Angles have the same measure if they can be superimposed on each other, which is $\angle A = \angle B$ iff $m\angle A = m\angle B$, in which case we say that $\angle A$ is **congruent** to $\angle B$. [2]
Definition: Given $\triangle ABC$ and $\triangle DEF$, if SAS, then $\triangle ABC \cong \triangle DEF$, which is saying, if $AB \cong DE$, $\angle A \cong \angle D$, and $AC \cong DF$, then $\angle B \cong \angle E$, $BC \cong EF$, and $\angle C \cong \angle F$. [2]
Definition: Given an angle $\angle ABC$ and a straight line $\overline{BC}$, with the points A and D on the

same side of BC, we say that $\angle ABC > \angle DBC$ if BD is between BA and BC. [1]

2.2 Theorems

With just these axioms we can prove the following theorems.

Theorem 2.1. *Vertical angles are always congruent. [3]*

Theorem 2.2. *If two sides of a triangle are congruent, then the angles opposite to these sides are congruent. [3]*

Theorem 2.3. *The measure of an exterior angle of a triangle is greater than the measures of either of the remote interior angles.*

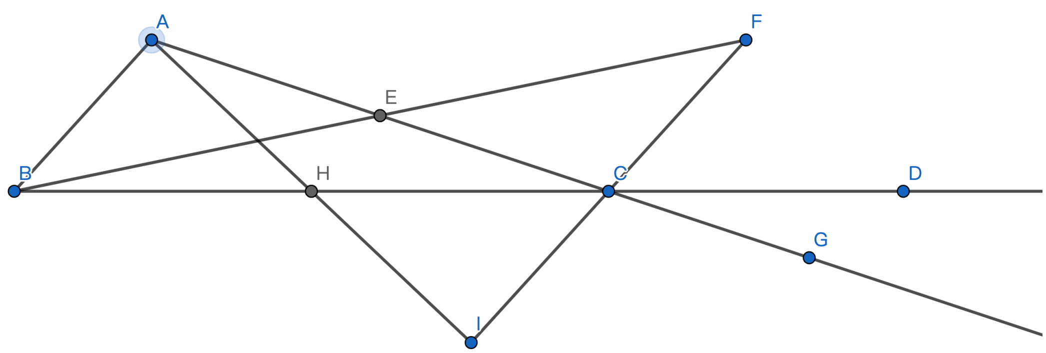

Figure 1: Theorem 2.3

Proof: In Figure 1, consider a $\triangle ABC$. Extend $\overline{BC}$ to D, pick the midpoint of $\overline{AC}$ as E, connect $\overline{BE}$, and extend $\overline{BE}$ to F such that $BE = FE$.
By Theorem 2.1, we know that $\angle AEB = \angle CEF$. So $\triangle AEB \cong \triangle CEF$. Then we have $AB = CF$, $\angle BAE = \angle FCE$. Therefore, $\angle DCE > \angle FCE = \angle BAE$. We use a similar technique to prove that $\angle DCE > \angle ABC$.
To prove that $\angle DCE > \angle ABC$, we construct a similar diagram by extending $\overline{AE}$ to G, picking the midpoint of $\overline{BC}$ as H, connecting $\overline{AH}$, and extending $\overline{AH}$ to I such that $AH = IH$. By Theorem 2.1, we know that $\angle BHA = \angle CHI$, $\angle DCE = \angle BCG$. So $\triangle BHA \cong \triangle CHI$. Then we have $AB = IC$, $\angle ABH = \angle ICH$. Therefore, $\angle DCE = \angle BCG > \angle ICH = \angle ABH$. Thus, the measure of an exterior angle of a triangle is greater than the measure of either of the remote interior angles. $\square$

The angle sum of a triangle is an important property that differentiates between Euclidean Geometry and non-Euclidean Geometry. Here is the best we can do in Neutral Geometry.

Theorem 2.4. *The sum of the measures of the interior angles of a triangle is at most 180°.*

Wallance proved this theorem by contradiction. The proof is too tedious so we will not restate it here.

Theorem 2.5. *If a transversal falls on the lines such that a pair of alternate interior angles are congruent, then the lines are parallel.*

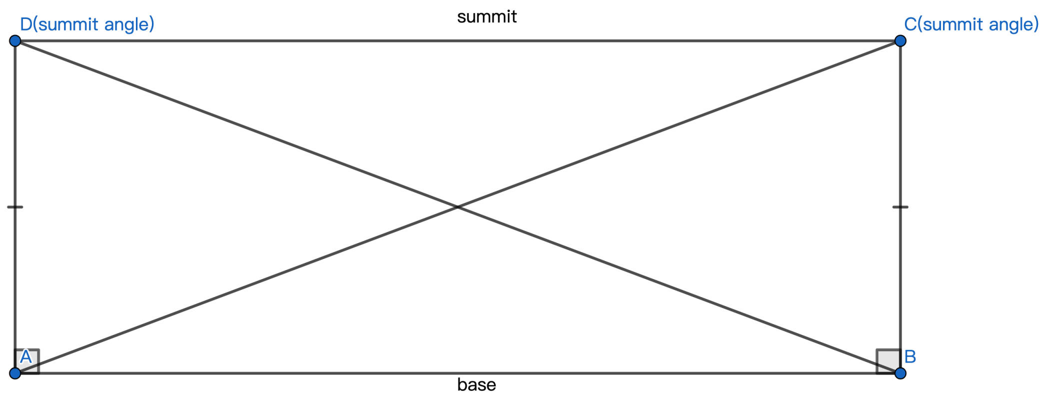

Figure 2: Definition: Saccheri Quadrilateral

Definition: A **Saccheri quadrilateral** is a quadrilateral in which a pair of opposite sides are equal and have one of the other sides as a common perpendicular (see Figure 2). The common perpendicular is the **base**, the side opposite to it is the **summit**, and the angles adjacent to the summit are the **summit angles**. [1]

The idea of a Saccheri quadrilateral is crucial to understanding the difference between Euclidean and non-Euclidean Geometry.

Theorem 2.6. *The summit angles of a Saccheri quadrilateral are congruent, and the line jointing the midpoints of the bases is perpendicular to both bases.*

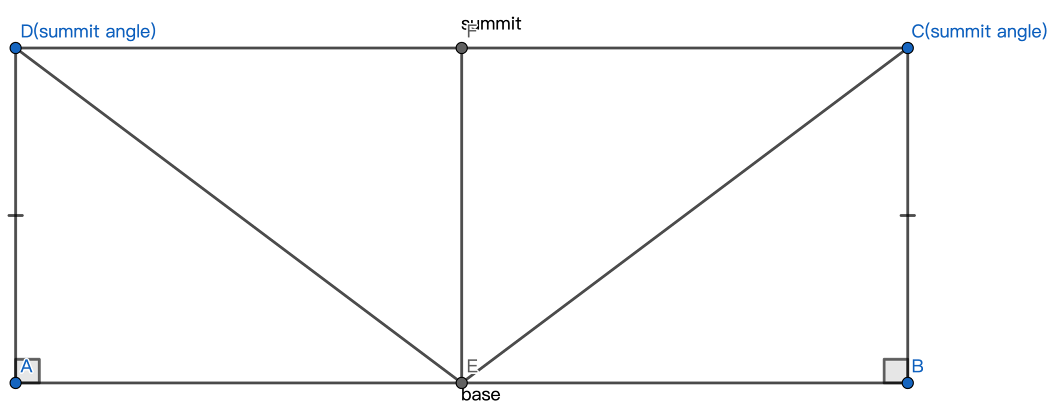

Figure 3: Minpoint Segment Perpendicular

Proof: Suppose we have a Saccheri quadrilateral $ABCD$ like figure 3. Connecting $\overline{BD}$ and $\overline{AC}$ (Figure 2), we get $\triangle ABD \cong \triangle BAC$ by SAS, so $BD = AC$, $\angle ADB = \angle BCA$ and $\angle ABD = \angle BAC$. Then we get $\angle DAC = \angle CBD$. So $\triangle DAC \cong \triangle CBD$ by SAS. Therefore, $\angle ADC = \angle BCD$.
Furthermore, we can show that the line jointing the midpoints of the bases is perpendicular to both bases. In Figure 3, let E and F be the mid points of $\overline{AB}$ and $\overline{CD}$. Connect $\overline{EF}$, $\overline{DE}$ and $\overline{CE}$. By SAS, we get $\triangle AED \cong \triangle BEC$. So $DE = CE$ and $\angle ADE = \angle BCE$. We know that $\angle ADC = \angle BCD$ from above. So again by SAS, $\triangle DEF \cong \triangle CEF$. Since F is a point on DC and $\angle DFE = \angle CFE$ are a linear pair, so $\angle DFE = \angle CFE = 90°$. Similarly, we can get $\angle AEF = \angle BEF = 90°$ by connecting $\overline{AF}$ and $\overline{BF}$. Thus, the summit angles of a Saccheri quadrilateral are congruent. In addition, the line jointing the midpoints of the bases is perpendicular to both bases.

The following theorem helps us understanding the difference between Euclidean and non-Euclidean Geometry.

Theorem 2.7. *The summit angles of a Saccheri quadrilateral are not obtuse and thus are both acute or both right. [3]*

Here we are not going to prove Theorem 2.7; instead, we will prove a weaker result. However, before that, we still need several extra theorems.

Definition: $A - B - C$ means A, B and C are collinear and B is between A and C. [2]

Theorem 2.8. *If, from the endpoints of a given side of a $\triangle ABC$, we draw perpendiculars $\overline{BF}$ and $\overline{CG}$ to a straight line through the midpoints D and E of the other two sides, $\overline{AB}$ and $\overline{AC}$, forming a quadrilateral $GCBF$, then the following are true.*
(a) The quadrilateral is a Saccheri quadrilateral whose summit is the given side $\overline{BC}$ of the triangle.
(b) The base $\overline{FG}$ is twice the length of the straight line $\overline{DE}$ joining the midpoints of the triangles other two sides.
(c) Its two summit angles ($\angle FBC$ and $\angle GCB$) have the same sum as the three interior angles of the triangle.

Proof: Let H be the foot of a $\perp$ from A to $\overline{DE}$. Then either $H - D - E$, $D - H - E$, $D - E - H$, $D = H$ or $E = H$. We first consider the situation when $D - H - E$, see Figure 4. We know $D - H - E$, $B - D - A$. If H and F are on the same side of $\overline{AB}$, by the exterior angle theorem, since $\angle AHE$ is right, we get $\angle ADH$ is acute and $\angle BDH$ is obtuse. Then $\angle BFH > \angle BDH$, by the exterior angle theorem, this is a contradiction. So H and F are on the opposite sides of $\overline{AB}$, so we have $F - D - H$.
Now, $BD = AD$, $\angle BDF \cong \angle ADH$, $\angle BFD \cong \angle AHD$, so $\triangle BDF \cong \triangle ADH$ by AAS. Also, $AE = CE$, $\angle AHE \cong \angle CGE$, $\angle AEH \cong \angle CEG$, so $\triangle AHE \cong \triangle CGE$ by AAS. Thus, $FB = AH = GC$. Therefore, $FBCG$ is a Saccheri quadrilateral. This proves (a).

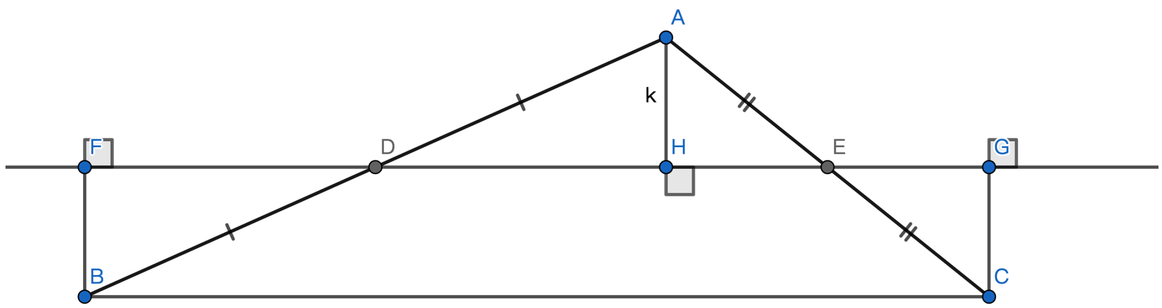

Figure 4: Triangle and Saccheri Quadrilateral

Also, $FD = DH$ and $HE = GE$, so

$$\begin{aligned} FG &= FD + DH + HE + EG \\ &= DH + DH + HE + HE \\ &= 2(DE + HE) \\ &= 2DE. \end{aligned}$$

This proves (b)
Plus, $\angle DBF = \angle DAH$ and $\angle GCE = \angle EAH$, so

$$\begin{aligned} \angle A + \angle B + \angle C &= \angle DAH + \angle EAH + \angle ABC + \angle ACB \\ &= \angle DBF + \angle GCE + \angle ABC + \angle ACB \\ &= \angle FBC + \angle GCB. \end{aligned}$$

This proves (c).
The other four situations are similar so we omit the proof here. $\square$

Theorem 2.9. *The line segment connecting the midpoints of two sides of a triangle is parallel to the third side.*

Proof: In Figure 5, consider a $\triangle ABC$, pick the midpoint of $\overline{AB}$ as D, pick the midpoint of $\overline{AC}$ as E, connect DE and extend it to F such that $DE = EF$, connect $\overline{CF}$. By Theorem 2.1, we know that $\angle AED = \angle CEF$. So $\triangle AED \cong \triangle CEF$. Then we get the alternate interior angles $\angle CFE$ and $\angle ADE$ are congruent. So $\overline{CF} \parallel \overline{AB}$.
In addition, we have $CF = AD = DB$. Now we have a $\square BCFD$, so $\overline{DE} \parallel \overline{BC}$. Thus, the line segment connecting the midpoints of two sides of a triangle is parallel to the third side. $\square$
Definition: Equidistant means every point on one line is the same distance from the other. [1]

Equidistance is another property that distinguishes Euclidean and non-Euclidean Geometry. Theorem 2.10 is the best we can do in Neutral Geometry.

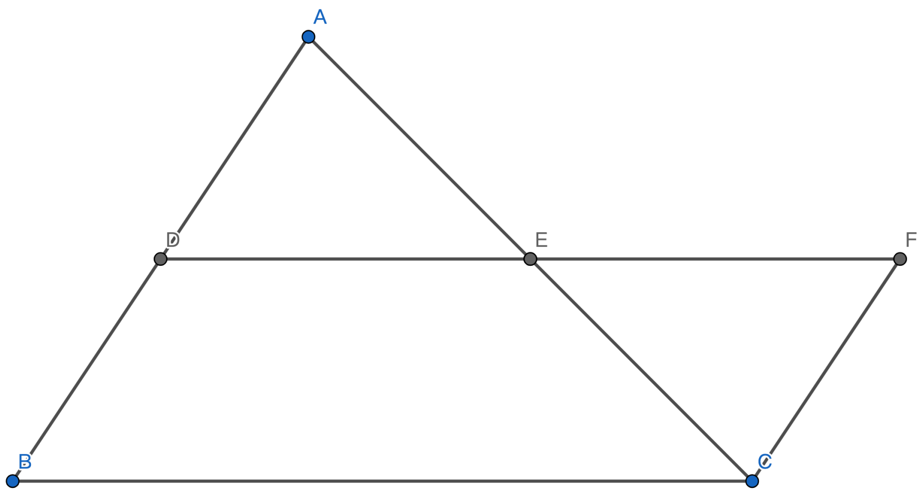

Figure 5: Theorem 2.9

Theorem 2.10. *Let l and m have a common perpendicular. That is, suppose $P, S \in l$ and $Q, T \in m$ are such that $\overline{PQ}$ and $\overline{ST}$ are perpendicular to both l and m. Then l and m are equidistant. [2]*

Now we have had sufficient theorems we need to prove the weaker version of Theorem 2.7 and here is it.

Theorem 2.11. *If there exists one Saccheri quadrilateral whose upper base angles are right, then the upper base angles of every Saccheri quadrilateral are right.*

Proof: Let $ABCD$ be Saccheri quadrilateral. Assume $\angle B$ and $\angle C$ are right. Let $XYZW$ be a Saccheri quadrilateral. Let M be the midpoint of $\overline{AD}$ and N be the midpoint of $\overline{BC}$. By Theorem 2.6 and Theorem 2.10, $MN = CD$. Now we need to consider 3 cases, $MN < XY$, $MN = XY$ and $MN > XY$. We will prove the situation where $MN < XY$ first. Suppose $MN < XY$,
See Figure 6, let $P \in MN$ be such that P is on the same of $\overline{AD}$ as N and such that $PM = XY$. Let $PS \perp MM$ at S, and let Q be such that $PQ = XW$. Let T be the foot of a perpendicular from Q to $\overline{AD}$. Since $\overline{PQ} \parallel \overline{BC} \parallel \overline{AD}$, Q, T are on opposite sides of $\overline{BC}$. Therefore, we get $\overline{BC}$ will intersect $\overline{QT}$ at some point V. By Theorem 2.10, $VT = CD$, so $CVTD$ is a rectangle and $\angle CVT = \pi/2$. Since $\overline{PM}$ and $\overline{QT}$ have two common perpendiculars, they are equidistant by Theorem 2.10. So $PQ = MT$. Thus, $\angle PQT = \angle MTQ = \pi/2$ by Theorem 2.6. So $PM = QT$. Thus, $PMTQ$ is a Saccheri quadrilateral. So $PMTQ$ and $XYZW$ are both Saccheri quadrilateral, with $PM = XY$, $TQ = ZW$ and $PQ = XW$. So $\angle M = \angle Y = \angle Z = \pi/2$. $\square$

Theorem 2.12. *The summit angles of a Saccheri quadrilateral are not obtuse and thus are both acute or both right.*

This is an important theorem but we are not going to prove it here.

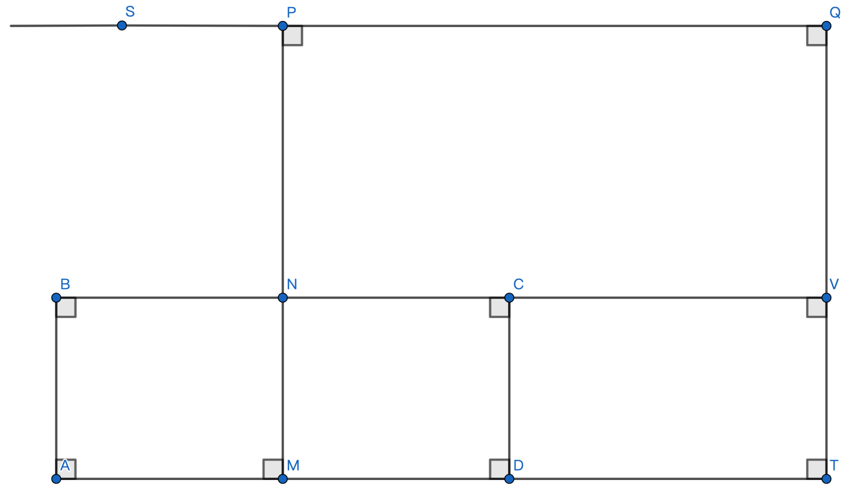

Figure 6: Theorem 2.10

Definition: A quadrilateral having four right angles is a **rectangle**. [2]

Theorem 2.13. *If one rectangle exists, then every triangle has an angle sum of 180°. [3]*

Proof: By Theorem 2.8(c), we know that two summit angles of a Saccheri quadrilateral have the same sum as the three interior angles of the triangle. Suppose one rectangle exists, by Theorem 2.11, all Saccheri quadrilaterals are rectangles. Then the sum of three interior angles of the triangle equals to two right angles, which is 180°. □

Neutral geometries are somewhat clear and straightforward so we only gave proof for some selected theorems above. Later in the paper, we will see more proofs based on the proofs in neutral geometry. After all, both Euclidean geometry and hyperbolic geometry use the four axioms in neutral geometry as their first four axioms.

3 Euclidean Geometry

3.1 Axioms

Adding the parallel postulate to the four axioms of neutral geometry gives us the axioms of Euclidean geometry.
Axiom 1. We may draw a straight line between any two points.
Axiom 2. We may extend any terminated straight line indefinitely.
Axiom 3. We may draw a circle with any given point as center and any given radius.
Axiom 4. All right angles are equal.

The first 4 axioms are the same as in Neutral Geometry. For Euclidean Geometry, we add the following axiom.

Axiom 5. Given a line and a point not on the line, only one unique line through the point is parallel to the given line

3.2 Theorems

Theorem 3.1. *If a transversal crosses two parallel lines, the alternate interior angles are congruent.*

Theorem 3.2. *The measure of an exterior angle is equal to the sum of the measures of the two remote interior angles of the triangle.*

Theorem 3.1 directly follows from the converse of the alternate interior angle theorem. In addition, Theorem 3.2 is a direct consequence of Theorem 3.1. Theorem 3.2 is an example which theorem is true in Euclidean Geometry since it is true in Neutral Geometry. However, there are some theorems that are true in Euclidean Geometry while are not true in Neutral Geometry, because it relies on the parallel postulate. Theorem 3.3 is an example for this.

Theorem 3.3. *The sum of the measures of the interior angles of a triangle is 180°.*

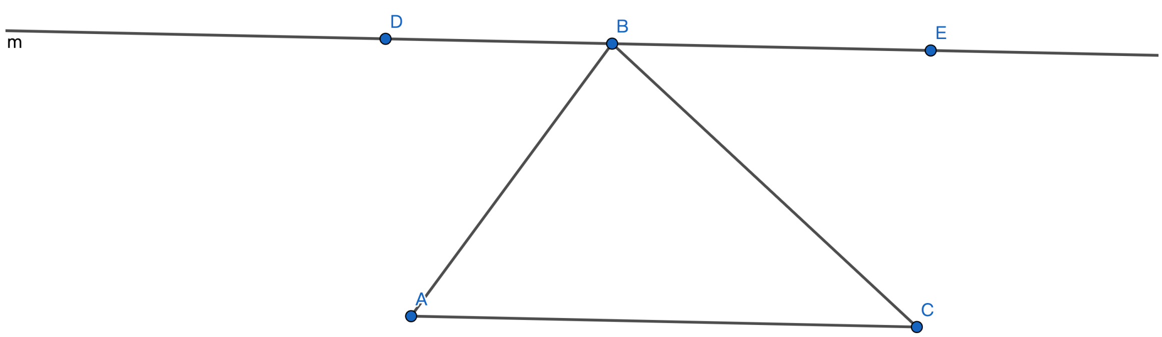

Figure 7: Theorem 3.3

Proof: Consider $\triangle ABC$ shown in Figure 7. According to the Euclidean parallel postulate, there is a unique line m through B parallel to line $\overline{AC}$. Pick two points D, E on line m such that $D - B - E$. Since $\angle ABD$, $\angle ABC$ and $\angle CBE$ form a linear triple, their sum is 180°. Applying the converse of the alternate interior triangle theorem, we see that $m\angle ABD = m\angle BAC$ and $m\angle CBE = m\angle ACB$. Since $m\angle ABD + m\angle ABC + m\angle CBE = 180°$. We have, by substitution, $m\angle BAC + m\angle ABC + m\angle ACB = 180°$. $\square$

Theorem 3.4. *The line segment connecting the midpoints of two sides of a triangle is congruent to half of the third side.*

Proof: Consider Theorem 2.9 and Figure 5. We have shown BCFD is a parallelogram and $\triangle AED \cong \triangle CEF$. So $DF = BC, DE = EF$. Therefore, $DE = (1/2) * BC$. □

The three theorems (Theorem 3.2, Theorem 3.3 and Theorem 3.4) above had a similar version in neutral geometry. However, in order to get a more specific conclusion, we have to prove them in Euclidean geometry, since we used the parallel axiom. Here are some other theorems in Euclidean geometry that are quite valuable for our research.

Theorem 3.5. *Rectangles exist in Euclidean geometry. [3]*

Theorem 3.6. *Parallel lines are everywhere equidistant in Euclidean geometry.*

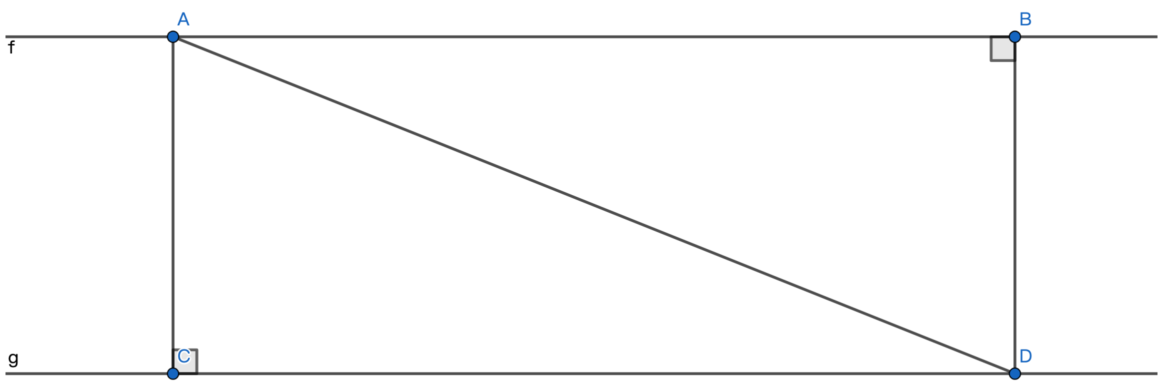

Figure 8: Theorem 3.7

Proof: Consider two parallel lines $f \parallel g$. As Figure 8 shows, on f, pick a point A and draw $\overline{AC} \perp g$, intersecting g at point C. On g, pick a point D and draw $\overline{BD} \perp f$, intersecting f at point B.
Then we have $\angle ACD = \angle DBA$, $\angle ADC = \angle DAB$, $AD = DA$.
By AAS, $\triangle ACD \cong \triangle DBA$. So $\angle CAD = \angle BDA$, $\angle CDA = \angle BAD$.
Since sum of angles of a triangle is 180°, $\angle CAB = \angle BDC = 90°$. Therefore, $ACDB$ is a rectangle. This proves Theorem 3.5. In addition, $AC = DB$, this proves Theorem 3.6
Thus, rectangles exist in Euclidean geometry. In addition, parallel lines are everywhere equidistant in Euclidean geometry. □

Those two theorems are typical examples of theorems that are true in Euclidean geometry while their negation is true in hyperbolic geometry. We prove them soon.

4 Hyperbolic Geometry

4.1 Gauss and Bolyai

Gauss left the task of publishing results in non-Euclidean geometries to other mathematicians. One of individuals who rose to this challenge was the Hungarian mathematician Janos Bolyai. Janos was the son of Farkas Bolyai, who worked with Gauss on a variety of projects.

During the early 1820s, while studying in Vienna, Bolyai began to work on what he called a “complete system of non-Euclidean geometry." In fact, Bolyai was developing much of what today we call hyperbolic geometry. Farkas Bolyai published his son’s work as an appendix to one of his own essays. When Gauss encountered this appendix, he replied, “To praise this work would amount to praising myself. For the entire content of the work... coincides almost exactly with my own meditations."
These remarks discouraged Janos Bolyai. Although he produced over 20,000 pages of manuscript in mathematics, his only publication was the 24-page paper on non-Euclidean geometry. [3]

4.2 The Negation of the Parallel Postulate

There are two different cases if we take the negation of the Euclidean parallel postulate. One of them is “Given a line and a point not on the line, no line through the point is parallel to the given line" while the other one is “Given a line and a point not on the line, at least two lines through the point are parallel to the given line". If we consider the first negation, we get spherical geometry, while it gives us the hyperbolic geometry if we consider the second negation. Furthermore, if there are at least two lines through the point are parallel to the given line, there are actually infinitely many lines through the point are parallel to the given line. So we choose to use this negation as the fifth axiom of hyperbolic geometry, which is “Given a line and a point not on the line, infinitely many lines through the point are parallel to the given line".

4.3 Axioms

Adding this to the few axioms of neutral geometry gives us the axioms of hyperbolic geometry.
Axiom 1. We may draw a straight line between any two points.
Axiom 2. We may extend any terminated straight line indefinitely.
Axiom 3. We may draw a circle with any given point as center and any given radius.
Axiom 4. All right angles are equal.

The first 4 axioms are the same as in Neutral Geometry. For Hyperbolic Geometry, we add the following axiom.

Axiom 5. Given a line and a point not on the line, infinitely many lines through the point are parallel to the given line.

Theorem 4.1. *The summit angles of a Saccheri quadrilateral are acute.*

Proof: Assume that the summit angles of a Saccheri quadrilateral are not acute. Since we have Theorem 2.7 and Theorem 2.12, we can get a result that the summit angles are both right angles. Now we have found a rectangle, then by Theorem 2.13 every triangle has an angle sum of $180°$. This is equivalent to the Euclidean parallel postulate. This is a contradiction for hyperbolic parallel postulate. Therefore, the summit angles of a Saccheri quadrilateral are acute. □

The Italian mathematician Giovanni Girolamo Saccheri in 1733 tried to prove the summit angles of a Saccheri quadrilateral are 90° but failed. However, in the mid-19th century, Eugenio Beltrami, rediscovered this, as a significant result on non-Euclidean geometry.

In the proof of Theorem 4.1, we have actually proved the following theorem.

Theorem 4.2. *Rectangles do not exist in hyperbolic geometry.*

Theorem 4.3. *Parallel Lines are not everywhere equidistant in hyperbolic geometry.*

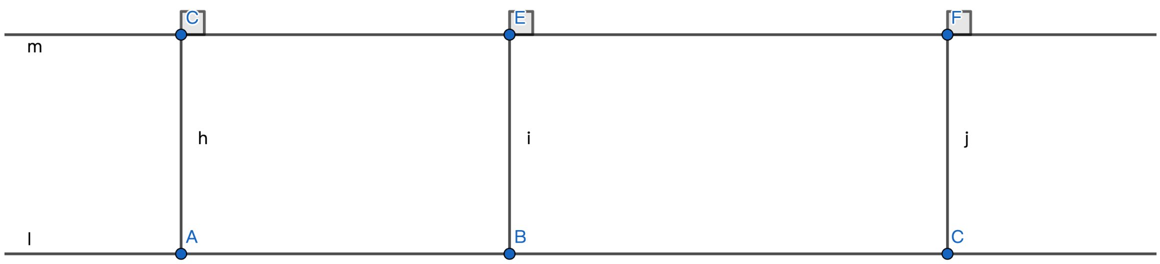

Figure 9: Theorem 4.3

Proof: To prove this, we must show that, if given any pair of parallel lines l and m, we can find at least one pair of points on line l for which the distance to m are not the same. To prove this, we will begin with any three points on l and proceed in the following way. In Figure 9, choose any pair of parallel lines l and m and any three distinct points A, B and C on l such that $A - B - C$ (A, B and C are collinear pairs). D, E and F are the feet of the perpendicular, respectively, from A, B and C to m.
Assume $AD = BE = CF$. Then Figure 9 displays three "inverted" Saccheri quadrilaterals: $\square DABE$, $\square EBCF$ and $\square DACF$. So $m\angle DAB = m\angle EBA$, $m\angle EBC = m\angle FCB$, and $m\angle DAB = m\angle FCB$. Then we have the congruent linear pair $m\angle EBA = m\angle EBC = 90°$. Therefore, $m\angle EBC = m\angle FCB = m\angle DAB = m\angle FCB = 90°$. Therefore, $\square DABE$, $\square EBCF$ and $\square DACF$ are all rectangles. However, this a contradiction by Theorem 4.2. Therefore, $AD = BE = CF$ is false and we can conclude that parallel lines are not everywhere equidistant in hyperbolic geometry. □

Here we gave a proof to the statement we made in the last section. Some theorems are true in Euclidean geometry, while its negation is true in hyperbolic geometry.

Theorem 4.4. *The sum of the measures of the interior angles of a triangle is less than 180°.*

Proof: Let $\triangle ABC$ be any triangle, see Figure 5. Bisect $\overline{AB}$ at D and $\overline{AC}$ at E. Connect $\overline{DE}$ and extend $\overline{DE}$ towards both directions. Draw $\overline{BF}$ and $\overline{CG}$ $\perp$ to the extended $\overline{DE}$. Then we get a Saccheri quadrilateral $GFBC$ with summit $\overline{BC}$ by Theorem 2.8. Draw $\overline{AH} \perp \overline{DE}$. Again by Theorem 2.8, $\angle FBC + \angle GCB =$ the angle sum of $\triangle ABC$. By Theorem 4.1, $\angle FBC < 90°$, $\angle GCB < 90°$. Therefore, $\angle FBC + \angle GCB$ is less than 180°. Thus, the sum of the measures of the interior angles of a triangle is less than 180°. □

Comparing with similar theorems in the other two geometry systems, this theorem gives us an excellent example that small changes to even one axiom in a geometry system can lead us to a very different conclusion.

5 Acknowledgments

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